ON THE ROOTS OF THE RIEMANN ZETA FUNCTION*

BY

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The object of the following paper is to simplify the methods and formulas developed and used by Gram,† Lindelöf,‡ and Backlund,§ in numerical investigations connected with the roots of the Zeta function. I apply these to locating and calculating additional roots.

I start with the formulas, as given by Backlund,

$$\zeta(s) = \sum_{\nu=1}^{n-1} \nu^{-s} + \frac{1}{2} n^{-s} + \frac{n^{1-s}}{s-1} + \sum_{\nu=1}^{k} T_\nu + R_k, \tag{1}$$

$$T_\nu = (-1)^{\nu-1} \frac{B_\nu}{(2\nu)!} \frac{s(s+1)\cdots(s+2\nu-2)}{n^{s+2\nu-1}}, \tag{2}$$

$$|R_k| < \frac{|s+2k+1|}{\sigma+2k+1} |T_{k+1}|, \qquad s = \sigma + ti, \tag{3}$$

and, when $\sigma = \frac{1}{2}$,

$$\begin{aligned} |T_\nu| &= \frac{\alpha_\nu}{\sqrt{n}} \left(\frac{t}{n}\right)^{2\nu-1}, \\ \alpha_\nu &= \frac{B_\nu}{(2\nu)!} \sqrt{\left(1+\frac{1}{4t^2}\right)\left(1+\frac{9}{4t^2}\right)\cdots\left(1+\frac{(4\nu-3)^2}{4t^2}\right)}. \end{aligned} \tag{4}$$

Formula (3) often gives too large an upper limit for $|R_k|$. A smaller limit can generally be obtained, as suggested by Lindelöf, by using

$$|R_k| < |T_{k+1}| + |T_{k+2}| + \cdots + |T_l| + |R_l|, \tag{5}$$

with a suitable choice for l. To calculate an upper limit for a given remainder with any degree of precision by means of (3) and (5) is quite laborious. To shorten the work, determine the ratio of $|T_{\nu+1}|$ to $|T_\nu|$ by means of (4):

$$\left|\frac{T_{\nu+1}}{T_\nu}\right| = \frac{B_{\nu+1}}{B_\nu} \left(\frac{t}{n}\right)^2 \frac{\sqrt{\left[1+\left(\frac{4\nu-1}{2t}\right)^2\right]\left[1+\left(\frac{4\nu+1}{2t}\right)^2\right]}}{(2\nu+1)(2\nu+2)}. \tag{6}$$

* Presented to the Society, October 25, 1924.

† *Note sur les zéros de la fonction $\zeta(s)$ de Riemann*, Acta Mathematica, vol. 27 (1903), p. 289.

‡ *Sur une formule sommatoire générale,* Acta Mathematica, vol. 27, p. 305.

§ R. J. Backlund, *Ueber die Nullstellen der Riemannschen Zetafunktion*, Dissertation, Helsingfors, 1916.

Consider the identity

$$\left[1+\left(\frac{4\nu-1}{2t}\right)^2\right]\left[1+\left(\frac{4\nu+1}{2t}\right)^2\right] = \left[1+\frac{16\nu^2+1}{4t^2}\right]^2-\left(\frac{2\nu}{t^2}\right)^2.$$

If t is large in comparison with ν, as will be the case in what follows, the last term is very small and may be omitted. The error thus introduced does not ordinarily affect the first seven decimal places. This gives in place of (6) the much simpler approximate formula

$$(7) \qquad |T_{\nu+1}| = b_\nu\left[1+\frac{(4\nu)^2+1}{4t^2}\right]\left(\frac{t}{n}\right)^2 |T_\nu|,$$

in which

$$b_1 = \frac{1}{60}, \quad b_2 = \frac{1}{42}, \quad b_3 = \frac{1}{40}, \quad b_4 = \frac{10}{396},$$

$$\begin{aligned} b_5 &= .025\,311\,355, \\ b_6 &= .025\,325\,615, \\ b_7 &= .025\,329\,132, \\ b_8 &= .025\,330\,005\,5, \\ b_9 &= .025\,330\,223, \\ b_{10} &= .025\,330\,278, \\ b_{11} &= .025\,330\,291, \\ b_{12} &= .025\,330\,295, \\ b_{13} &= .025\,330\,296. \end{aligned}$$

These coefficients are evidently converging to a limit. In fact if we use in

$$b_\nu = \frac{B_{\nu+1}}{B_\nu}\cdot\frac{1}{(2\nu+1)(2\nu+2)}$$

the relations

$$B_\mu = \frac{(2\mu)!}{2^{2\mu-1}\pi^{2\mu}}\cdot\zeta(2\mu), \qquad \lim_{\mu\to\infty}\zeta(2\mu) = 1,$$

we obtain

$$\lim_{\nu\to\infty} b_\nu = \frac{1}{4\pi^2} = .025\,330\,295\,91.$$

To formula (7) should be joined the first formula (4), namely

$$(7') \qquad |T_1| = \frac{\sqrt{t^2+\frac{1}{4}}}{12\,n^{3/2}}.$$

It is desirable to have some simple method for determining the value for l in (5) that will give the lowest upper limit for $|R_k|$. Suppose this limit determined from two consecutive values of l, $l = \lambda$ and $l = \lambda+1$, and assume that the right member of (5) is less for $l = \lambda+1$ than it is for $l = \lambda$. This leads to the inequality

$$|R_{\lambda+1}| + |T_{\lambda+1}| < |R_\lambda|,$$

in which $|R_\nu|$ is used to indicate, not the actual numerical value of the remainder, but its upper limit. Accordingly, replace $|R_{\lambda+1}|$ and $|R_\lambda|$ by the right members of (3) and then replace $|T_{\lambda+2}|$ by (7). Put $b_{\lambda+1} = .02533$ and divide out the common factor $|T_{\lambda+1}|$. In the resulting formula drop the three fractional terms having denominators $4t^2$, these being small in comparison with the other terms. The effect is to strengthen somewhat the inequality and we obtain the relation

$$\frac{.02533\,t}{2\lambda+\frac{7}{2}}\left(\frac{t}{n}\right)^2+1<\frac{t}{2\lambda+\frac{3}{2}}, \tag{8}$$

from which to determine the largest possible value of λ when t and n are given. The easiest way to find λ from (8) is by trial. If the value obtained makes both members very nearly equal, the next lower integer should be taken for λ, giving $l = \lambda+1$ as the best value to use in (5).

Suppose $t' > t$ and $n' > n$ are two other numbers such that

$$\frac{t'}{n'} = \frac{t}{n}. \tag{9}$$

Denote by T' and R' the new values of T and R. Then from (4) and (9) we deduce

$$|T'_\nu| < \sqrt{\frac{n}{n'}}\,|T_\nu|, \tag{10}$$

and from (3) and (10) follows

$$|R'_k| < \sqrt{\frac{n'}{n}}\,|R_k|. \tag{11}$$

Combining (5), (10), and (11) we obtain, finally, the very useful formula

$$|R'_k| < \sqrt{\frac{n}{n'}}\,|R_k| + \left[\sqrt{\frac{n'}{n}} - \sqrt{\frac{n}{n'}}\right]|R_l|. \tag{12}$$

The upper limit determined for $|R'_k|$ by (12) is of course applicable for any t between t and t' if n remains fixed at the value n'.

If we write $\zeta(s)$ in the form

$$\zeta(s) = \varrho e^{i\varphi} = \varrho \cos\varphi + i\varrho \sin\varphi \tag{13}$$

and take $s = \frac{1}{2} + ti$, then ϱ and φ are functions of t, the latter of which was obtained by Gram in a very simple approximate form. Following the example of Gram, I will denote the real and imaginary components of $\zeta(\frac{1}{2} + ti)$ by $C(t)$ and $S(t)$ respectively, so that

$$C(t) = \varrho(t)\cos\varphi(t), \qquad S(t) = \varrho(t)\sin\varphi(t).$$

Further, the roots of $\cos\varphi(t) = 0$ will be represented by β_n, the roots of $\sin\varphi(t) = 0$ by γ_n, and the roots of $\varrho(t)$, which are the roots of the Zeta function, by α_ν. Gram calculated the first fifteen of the roots α and called attention to the fact that the α's and the γ's separate each other. I will refer to this property of the roots as *Gram's Law*. Gram expressed the belief that this law is not a general one. It is one of the objects of this paper to establish that fact. For this purpose I make use of a theorem proved by Gram which states that if $C(t)$ takes the same sign when $t = \gamma_\nu$ and $t = \gamma_{\nu+1}$, then at least one root α occurs between these two values of t.

Taking the real terms in (1) with $\sigma = \frac{1}{2}$, we have

$$C(t) = K + C_0 + C_1 + \cdots + C_k + r_k, \tag{14}$$

in which

$$K = 1 + \sum_{\nu=2}^{n-1} \frac{1}{\sqrt{\nu}} \cos(t\log\nu) + \frac{1}{2\sqrt{n}}\cos(t\log n), \tag{15}$$

$$C_0 = C_0(t) = -\frac{\sqrt{n}}{t^2 + \frac{1}{4}}\left(ts + \frac{1}{2}c\right), \qquad \begin{aligned} s &= \sin(t\log n), \\ c &= \cos(t\log n), \end{aligned}$$

$$C_1 = \frac{ts + \frac{1}{2}c}{12\, n^{3/2}}, \tag{16}$$

$$C_2 = \frac{\left(ts + \frac{1}{2}c\right) + 4c}{720\, n^{3/2}}\left(\frac{t}{n}\right)^2 - \cdots,$$

(16)
$$C_3 = \frac{\left(ts+\frac{1}{2}c\right)+12c}{30240\,n^{3/2}}\left(\frac{t}{n}\right)^4 - \cdots,$$
$$C_4 = \frac{\left(ts+\frac{1}{2}c\right)+24c}{1209600\,n^{3/2}}\left(\frac{t}{n}\right)^6 - \cdots,$$
$$|r_k| \leqq |R_k|.$$

I use only the principal terms in C_2, C_3, and C_4 as the omitted terms do not affect the degree of accuracy required in this work.

To test Gram's Law, the values of γ_ν are calculated by the formula

$$\frac{t}{2\pi}\left(\log\frac{t}{2\pi}-1\right) = n+\frac{1}{8} \tag{17}$$

and substituted in (14)*. The value of t that satisfies (17) will be denoted by γ_ν, $\nu = n+3$.

As a result of the computations carried as far as $\gamma_{140} = 300.468$, it is found that $C(\gamma_\nu)$ *is positive in every case except two, viz.,* $\gamma_{129} = 282.455$, *and* $\gamma_{137} = 295.584$. As γ_{129} is the test case for Gram's Law, the value of $C(\gamma_{129})$ has been carefully verified, using $n = 100$ to obtain its value with great accuracy. Using C_4 as the last term and employing five decimals in the computations, the result is

$$C(\gamma_{129}) = -\ .027, \qquad |R_4| < .00005.$$

A question that naturally arises is this. The value of γ_{129} has been determined by an approximate formula (17). Is it possible that its exact value would make $C(t)$ positive? By calculation I find $S(282.455) = -\ .00015$ with $|R_4| < .00005$. It is obvious that the error in γ_{129} is very slight and that a correction in its value which would cause $S(t)$ to vanish could not change the sign of $C(t)$. For $t = \gamma_{137}$, we find $C(t) = -\ .017$.

To locate the roots α in these cases in which Gram's Law fails, it is necessary to get values of t which change the sign of $C(t)$. For $t = 282.6$ we find $C(t) = +\ .279$, which shows that $C(t)$ has a root between γ_{129} and 282.6. This is an α since the nearest root of $\cos\varphi = 0$ is $\beta_{126} = 283.28$. Since $C(t)$ has opposite signs at γ_{129} and γ_{130}, there must be an even number

* I am indebted to Dr. Jesse Osborne for carrying out most of these calculations. A Monroe calculating machine and the *Smithsonian Mathematical Tables* by Becker and Van Orstrand with the trigonometric functions of angles expressed in radian measure were indispensable adjuncts.

of roots α between these limits, according to Gram's theorem. Hence there must be two such roots at least, since one α has been found.

In like manner, since $C(295.4) = +.175$, there is a root α between $t = 295.4$ and γ_{137}, since the nearest β is $\beta_{133} = 294.76$. Hence there are two roots α, at least, in the interval $(\gamma_{136}, \gamma_{137})$. This gives the number of roots in the interval $(0, 300.468)$ as 138, at least, counting one root for each interval $(\gamma_\nu, \gamma_{\nu+1})$, with the exceptions just noted.

We now proceed to show that there are no other roots of $\zeta(s)$ in the region $0 < t < 300.468$. For this purpose the method of Backlund (with same modifications) is used. Let the number of zeros of $\zeta(s)$ for which $0 < t < T$ be denoted by $N(T)$. Then*

$$(18)\quad N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8} + \frac{1}{\pi}\Delta_{abc}\arg\zeta(s) + \frac{1}{\pi}R(T),$$

in which $\Delta_{abc}\arg\zeta(s)$ denotes the increment that $\arg\zeta(s)$ takes when s describes the broken line abc in the s-plane, starting at the point $a = \frac{3}{2}$ and moving along the straight line $\sigma = \frac{3}{2}$ to the point $b = \frac{3}{2} + iT$, thence along the straight line $t = T$ to the point $c = \frac{1}{2} + iT$. Moreover,

$$R(T) < \frac{1}{48T} + \frac{0.2}{T^3}.$$

Backlund proves that $\Delta_{abc}\arg\zeta(s)$ is numerically less than $\pi/2$, for the case $T = 200$, by proving that $\Re\zeta(s) = \varrho\cos\varphi$ does not vanish anywhere on the line abc. It follows that $\cos\varphi$ does not vanish at any point of this line and hence that φ does not pass through $\pm\pi/2$, assuming $\varphi = 0$ at a. The first step consists in proving that $\cos\varphi$ does not vanish on the line ab, T being entirely arbitrary. The second part of Backlund's proof, while very simple and ingenious, takes advantage of the fact that $\Re\zeta(\frac{1}{2} + 200i) = C(200)$ has an unusually large value, viz., 4.6. In the case I am dealing with, $T = 300.468$, we have $C(T) = 2.15$, which is too small for use in Backlund's method of proof. I accordingly proceed to modify the method so as to make it applicable to a much wider range of values of T.

On the line bc we have $s = \sigma + iT$,

$$(19)\qquad \frac{1}{2} \leq \sigma \leq \frac{3}{2}.$$

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* Backlund, p. 22.

Write $\Re\zeta(s)$ in the form

$$\Re\zeta(\sigma+iT) = K(\sigma)+L(\sigma), \tag{20}$$

$$K(\sigma) = \sum_{\nu=1}^{n-1} \nu^{-\sigma}\cos(T\log\nu)+\frac{1}{2}n^{-\sigma}\cos(T\log n), \tag{21}$$

$$L(\sigma) = \Re\left(\frac{n^{1-s}}{s-1}+\sum_{\nu=1}^{k} T_\nu+R_k\right). \tag{22}$$

Each term in (21) has the property of decreasing numerically as σ increases. Denote such a term, $\mu^{-\sigma}\cos(T\log\mu)$, by g_μ, if it is positive, and by h_μ, if it is negative. The signs of the individual terms do not change in the interval (19).

Consider, now, a sum of positive and negative terms, $G(\sigma)+H(\sigma)$,

$$G(\sigma) = g_{\mu_1}+g_{\mu_2}+\cdots+g_{\mu_i},$$
$$H(\sigma) = h_{\nu_1}+h_{\nu_2}+\cdots+h_{\nu_j},$$

in which the indices are subject to the inequalities

$$\mu_1<\mu_2<\cdots<\mu_i<\nu_1<\nu_2<\cdots<\nu_j. \tag{23}$$

Suppose further that the inequality

$$G(\sigma)+H(\sigma)>0 \tag{24}$$

is satisfied when $\sigma = \frac{1}{2}$. Then *relation* (24) *holds throughout the interval* (19). For $G(\sigma)$ evidently satisfies the inequalities

$$\mu_i^{1/2-\sigma}G\left(\frac{1}{2}\right)<G(\sigma)<\mu_1^{1/2-\sigma}G\left(\frac{1}{2}\right), \qquad \sigma>\frac{1}{2}.$$

There accordingly exists a number α, depending on σ, such that

$$G(\sigma) = \alpha G\left(\frac{1}{2}\right), \qquad \mu_1<\alpha^{1/(1/2-\sigma)}<\mu_i.$$

Similarly a number β exists such that

$$H(\sigma) = \beta H\left(\frac{1}{2}\right), \qquad \nu_1<\beta^{1/(1/2-\sigma)}<\nu_j,$$

whence, from (23), and since $\frac{1}{2}-\sigma<0$, follows $\beta<\alpha$. Hence we obtain the relation

$$G(\sigma)+H(\sigma) = \alpha\left[G\left(\frac{1}{2}\right)+H\left(\frac{1}{2}\right)\right]+(\beta-\alpha)H\left(\frac{1}{2}\right)>0.$$

Accordingly, if we can group the terms of $K(\frac{1}{2})$ into one or more sets of the form $G(\frac{1}{2})+H(\frac{1}{2})$ having the above properties and including all of the negative terms $h_\nu(\frac{1}{2})$, together with a sufficient number of positive terms $g_\mu(\frac{1}{2})$ to insure that each set is positive, then $K(\sigma)>0$ in the interval (19). If there are any unused positive terms of $K(\frac{1}{2})$, we endeavor to group them with negative terms occurring in $L(\frac{1}{2})$ so that each group shall be positive throughout (19).

Apply now to the case $T=\gamma_{140} = 300.468$, $n=51$. For the terms in $K(\frac{1}{2})$ I obtain the following results, in which the notation $(\mu_1, \mu_2, \cdots)$ means $g_{\mu_1}(\frac{1}{2})+g_{\mu_2}(\frac{1}{2})+\cdots$, while $-(\nu_1, \nu_2, \cdots)$ stands for $h_{\nu_1}(\frac{1}{2})+h_{\nu_2}(\frac{1}{2})+\cdots$;

$$(1, 2)-(3, 4, 6, 8) = +.235,$$

$$(5, 7, 9, 10)-(11, 13, 15, 16, 17, 20, 21) = +.130,$$

$$(12, 14, 18, 19, 22, 23, 24, 25, 26) -(27, 28, 30, 32, 34, 36, 37, 40, 41, 42) = +.114.$$

All the negative terms of $K=K(\frac{1}{2})$ have now been used.

The first term of $L(\sigma)$ is negative when $\sigma=\frac{1}{2}$. We easily deduce the inequality

$$\left|\Re\left(\frac{n^{1-s}}{s-1}\right)\right| = \left|\frac{(\sigma-1)c-Ts}{[(\sigma-1)^2+T^2]n^{\sigma-1}}\right|, \qquad c = \cos(T\log n), \quad s = \sin(T\log n),$$

$$< \frac{A}{n^\sigma}, \qquad A = \frac{\frac{1}{2}|c|+T|s|}{T^2}n,$$

which holds for all values of σ in (19). If we can find a sum of unused terms $G(\sigma)$ of $K(\sigma)$ such that $G(\sigma)-An^{-\sigma}>0$ for $\sigma=\frac{1}{2}$, then this inequality will hold throughout the interval (19). In the present case we find $An^{-1/2}=.004$ while $g_{29}(\frac{1}{2})=.183$, whence it follows that

$$g_{29}(\sigma)+\Re\left(\frac{n^{1-s}}{s-1}\right)>0$$

throughout (19).

For our present purposes it is unnecessary to discuss the remaining terms of (22), with the exception of the remainder which will be denoted by $R_k(\sigma)$. I find that a sufficient condition for the existence of the inequality

$$G(\sigma) + R_k(\sigma) > 0 \tag{25}$$

throughout (19) is

$$G\left(\frac{1}{2}\right) - Q_k\left|R_k\left(\frac{1}{2}\right)\right| > 0, \tag{26}$$

$$Q_k = \sqrt{\frac{T^2 + \left(2k + \frac{5}{2}\right)^2}{T^2 + \frac{1}{4}}}.$$

In the present case, taking $k = 0$, we find $|R_0(\frac{1}{2})| < .586$, while the sum of the remaining terms of $K(\frac{1}{2})$ is 1.492. As Q_k is obviously but slightly greater than 1, it is unnecessary to calculate its value to assure ourselves that (26) is abundantly satisfied and hence (25). We thus find that $\Re\zeta(s)$ is positive along the line bc. Hence $\Delta_{abc}\arg\zeta(s) < \pi/2$.

Returning to formula (18), we obtain the results

$$\frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) = 137.125,$$

$$\frac{1}{\pi}R(T) < .00003,$$

$$N(T) = 138 \pm \varepsilon,$$

$$\varepsilon = |\Delta_{abc}\arg\zeta(s) + R(T)| < .50003.$$

Since $N(T)$ is an integer, the only solution is $N(T) = 138$. As we have already located 138 roots α on the line $\sigma = \frac{1}{2}$, there are no other roots of $\zeta(s)$ in the region $0 < t < 300.468$.

If we wish to determine the number of roots in a larger interval, how shall we choose T without too much labor so that the above scheme is workable? Observation shows that $T = \gamma_\nu$ is likely to be a suitable choice. If by trial of the first terms of $C(\gamma_\nu)$ the choice of T is found unsuitable, the next adjacent γ is more than likely to answer the purpose. In the 121 cases in which all the terms of $C(\gamma_\nu)$ have been computed,

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there are 68 cases in which the proposed scheme is applicable. To see just how it works out in practise, I have tried it on the case $T = 500$. The nearest γ is $\gamma_{272} = 500.593$. We start out with the calculation of some of the initial terms of $C(t)$ and find them to be $1 + .114 - .568 - .474 + .065 + 007 + \cdots$. We already find that the excess of positive over negative terms has almost disappeared. Accordingly try $\gamma_{271} = 499.157$. The first terms start off so favorably that the calculation is continued to the end and it is readily found that all the terms including C_0 and R_0 can be arranged in positive groups in a way to insure that the function $\Re\zeta(\sigma + iT)$ will remain positive in the interval (19) and hence

$$\Delta_{abc} \arg \zeta(s) < \frac{\pi}{2}$$

for $T = \gamma_{271}$. From (18) we obtain the result: *The number of zeros of* $\zeta(s)$ *in the critical strip* $0 < t < 500$, $0 \leqq \sigma \leqq 1$ *is exactly* 269. This number satisfies the Riemann formula

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8}.$$

In fact this is exactly what the formula for the number of roots would become if we suppose that it is always possible to find a T, however large, such that $\Delta_{abc} \arg \zeta(s) < \pi/2$, since $R(T)$ in (18) is very small.

Gram has remarked on the strong tendency of $C(t)$ to take positive values and ascribes this to the fact that the series starts out with a large positive term $+1$. He expressed the belief, however, that the equilibrium would eventually be restored, and that $C(\gamma_\nu)$ would not always be positive. How slow $C(\gamma_\nu)$ has been to take a negative value, we have already seen. There is another law, observed by Dr. Osborne, which gives a still larger surplus in favor of the positive terms. *The series* $C(\gamma_\nu)$ *always has a group* G *of consecutive positive terms beginning with* ν (*the index of summation*) $= n_1$ *and ending with* $\nu = n_2$, *these integers increasing with* γ_ν *in such a way that the ratios* $\gamma_\nu : n_1$ *and* $\gamma_\nu : n_2$ *are very nearly constant, the first lying between* 7 *and* 8, *and the second between* 5 *and* 5.7. Moreover, the sum of the terms in each group is practically constant, being situated between 1.256 and 1.38. (In all except 9 cases this sum is greater than 1.32.) The number of terms in G gradually increases from 6, when $\gamma = 73.635$, to 12 in $C(\gamma_{140})$, and 16 in $C(\gamma_{271})$.

The following new roots of the Zeta function have been calculated by use of the series

$$S(t) = -\sum_{\nu=2}^{n-1} \frac{1}{\sqrt{\nu}} \sin(t \log \nu) - \frac{1}{2\sqrt{n}} \sin(t \log n)$$

$$+ \frac{\sqrt{n}}{t^2 + \frac{1}{4}} \left(\frac{1}{2} s - tc\right) + \frac{tc - \frac{1}{2} s}{12 n^{3/2}} + \frac{\left(tc - \frac{1}{2} s\right) - 4s}{720 n^{3/2}} \left(\frac{t}{n}\right)^2 + \cdots$$

$$+ \frac{\left(tc - \frac{1}{2} s\right) - 12s}{30240 n^{3/2}} \left(\frac{t}{n}\right)^4 + \cdots + \frac{\left(tc - \frac{1}{2} s\right) - 24s}{1209600 n^{3/2}} \left(\frac{t}{n}\right)^6 + \cdots,$$

$$s = \sin(t \log n), \qquad c = \cos(t \log n).$$

Only the principal terms of those derived from T_2, T_3, T_4 are retained, the parts omitted being too slight in value to affect the results. All of the calculations have been made with five decimals. The third decimal in α_n has been estimated by linear interpolation and may not be exact in all cases. I have recalculated the values of α_{11} to α_{15}, given by Gram to only one decimal. The results are as follows:

$\alpha_{11} = 52.970$,	$\alpha_{21} = 79.337$,
$\alpha_{12} = 56.446$,	$\alpha_{22} = 82.910$,
$\alpha_{13} = 59.347$,	$\alpha_{23} = 84.734$,
$\alpha_{14} = 60.833$,	$\alpha_{24} = 87.426$,
$\alpha_{15} = 65.113$,	$\alpha_{25} = 88.809$,
$\alpha_{16} = 67.080$,	$\alpha_{26} = 92.494$,
$\alpha_{17} = 69.546$,	$\alpha_{27} = 94.651$,
$\alpha_{18} = 72.067$,	$\alpha_{28} = 95.871$,
$\alpha_{19} = 75.705$,	$\alpha_{29} = 98.831$.
$\alpha_{20} = 77.145$,	

In finding a first approximation to a required α the following observed law has been very useful. If α lies on the segment from γ_ν to $\gamma_{\nu+1}$, it divides this into two segments $\gamma_\nu \alpha$ and $\alpha \gamma_{\nu+1}$ such that the ratio of the first to the second is > 1 (or < 1) according as the ratio $C(\gamma_\nu) : C(\gamma_{\nu+1})$ is > 1 (or < 1). Moreover, the first ratio is large or small according as the second ratio is large or small.

The following table gives the values of $C(\gamma_\nu)$ in the interval $200 < t < 300$. This table, in conjunction with that published by Backlund, locates all roots α in the interval $0 < t < 300$. The values of $C(\gamma_\nu)$ were calculated solely for the purpose of determining their signs and hence their values may not be very exact. A plus sign is used to indicate a large positive value in those cases in which it was unnecessary to complete the computation.

There is one root α situated between two consecutive values of γ_ν given in the table, with exception of the four cases discussed in the text in which one α lies in each of the intervals (282.455, 282.6), (282.6, 284.1), (294.0, 295.4), (295.4, 295.6).

ν	γ_ν	$C(\gamma_\nu)$	ν	γ_ν	$C(\gamma_\nu)$
82	201.5	0.6	112	254.0	3.7
83	203.3	1.2	113	255.7	0.9
84	205.1	0.6	114	257.4	2.6
85	206.9	3.6	115	259.1	0.8
86	208.7	2.0	116	260.8	0.1+
87	210.5	2.5	117	262.5	+
88	212.3	1.2	118	264.1	2.4
89	214.0	0.5	119	265.8	0.6
90	215.8	1.4	120	267.5	0.6
91	217.6	5.8	121	269.2	2.7
92	219.4	1.2	122	270.8	2.1
93	221.1	0.3	123	272.5	+
94	222.9	2.9	124	274.2	+
95	224.6	0.7	125	275.8	0.5
96	226.4	3.9	126	277.5	1.5
97	228.2	2.6	127	279.1	0.2
98	229.9	1.5	128	280.8	+
99	231.6	0.3	129	282.5	—0.027
100	233.4	0.9	130	284.1	1.4
101	235.1	5.4	131	285.8	1.9
102	236.9	0.8	132	287.4	1.0
103	238.6	1.5	133	289.0	1.9
104	240.3	1.3	134	290.7	4.3
105	242.0	1.7	135	292.3	1.6
106	243.8	0.9	136	294.0	0.8
107	245.5	+	137	295.6	—0.017
108	247.2	0.05+	138	297.2	3.4
109	248.9	0.8	139	298.8	3.2
110	250.6	0.8	140	300.5	2.2
111	252.3	3.2			

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